

# Local Spectrum of a Family of Operators

Simona Macovei\*

**Abstract.** Starting from the classic definitions of resolvent set and spectrum of a linear bounded operator on a Banach space, we introduce the local resolvent set and local spectrum, the local spectral space and the single-valued extension property of a family of linear bounded operators on a Banach space. Keeping the analogy with the classic case, we extend some of the known results from the case of a linear bounded operator to the case of a family of linear bounded operators on a Banach space.

## 1 Introduction

Let  $X$  be a complex Banach space and  $B(X)$  the Banach algebra of linear bounded operators on  $X$ . Let  $T$  be a linear bounded operator on  $X$ . The *norm* of  $T$  is

$$\|T\| = \sup \{\|Tx\| \mid x \in X, \|x\| \leq 1\}.$$

The *spectrum* of an operator  $T \in B(X)$  is defined as the set

$$Sp(T) = \mathbb{C} \setminus r(T),$$

where  $r(T)$  is the *resolvent set* of  $T$  and consists in all complex numbers  $\lambda \in \mathbb{C}$  for which the operator  $\lambda I - T$  is bijective on  $X$ .

An operator  $T \in B(X)$  is said to have the *single-valued extension property* if for any analytic function  $f : D_f \rightarrow X$ , where  $D_f \subset \mathbb{C}$  is open, with  $(\lambda I - T)f(\lambda) \equiv 0$ , it results  $f(\lambda) \equiv 0$ .

For an operator  $T \in B(X)$  having the single-valued extension property and for  $x \in X$  we can consider the set  $r_T(x)$  of elements  $\lambda_0 \in \mathbb{C}$  such that there is an analytic function  $\lambda \mapsto x(\lambda)$  defined in a neighborhood of  $\lambda_0$  with values in  $X$ , which verifies  $(\lambda I - T)x(\lambda) \equiv x$ . The set  $r_T(x)$  is said *the local resolvent set of  $T$  at  $x$* , and the set  $Sp_T(x) = \mathbb{C} \setminus r_T(x)$  is called *the local spectrum of  $T$  at  $x$* .

An analytic function  $f_x : D_x \rightarrow X$ , where  $D_x \subset \mathbb{C}$  is open, is said the *analytic extension* of function  $\lambda \mapsto R(\lambda, T)x$  if  $r(T) \subset D_x$  and  $(\lambda I - T)f_x(\lambda) \equiv x$ .

If  $T$  has the single-valued extension property, then, for any  $x \in X$  there is a unique *maximal analytic extension* of function  $\lambda \mapsto R(\lambda, T)x : r_T(x) \rightarrow X$ , referred from now as  $x(\lambda)$ . Moreover,  $r_T(x)$  is an open set of  $\mathbb{C}$  and  $r(T) \subset r_T(x)$ .

Let

$$X_T(a) = \{x \in X \mid Sp_T(x) \subset a\}$$

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\*simonamacovei@yahoo.com

be the *local spectral space* of  $T$  for all sets  $a \subset \mathbb{C}$ . The space  $X_T(a)$  is a linear subspace (not necessary closed) of  $X$ .

Two operators  $T, S \in B(X)$  are *quasinilpotent equivalent* if

$$\lim_{n \rightarrow \infty} \left\| (T - S)^{[n]} \right\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left\| (S - T)^{[n]} \right\|^{\frac{1}{n}} = 0,$$

where  $(T - S)^{[n]} = \sum_{k=0}^n (-1)^{n-k} C_k^n T^k S^{n-k}$ , for any  $n \in \mathbb{N}$ .

The quasinilpotent equivalence relation is an equivalence relation (i.e. is reflexive, symmetric and transitive) on  $B(X)$ .

**Theorem 1.** *Let  $T, S \in B(X)$  be two quasinilpotent equivalent operators. Then*

- i)  $Sp(T) = Sp(S)$ ;
- ii)  $T$  has the single-valued extension property if and only if  $S$  has the single-valued extension property. Moreover,  $Sp_T(x) = Sp_S(x)$ .

For an easier understanding of the results from this paper, we recall some definitions and results introduced by author in "Spectrum of a Family of Operators" [6].

We say that two families of operators  $\{S_h\}, \{T_h\} \subset B(X)$ , with  $h \in (0, 1]$ , are *asymptotically equivalent* if

$$\lim_{h \rightarrow 0} \|S_h - T_h\| = 0.$$

Two families of operators  $\{S_h\}, \{T_h\} \subset B(X)$ , with  $h \in (0, 1]$ , are *asymptotically quasinilpotent (spectral) equivalent* if

$$\lim_{n \rightarrow \infty} \limsup_{h \rightarrow 0} \left\| (S_h - T_h)^{[n]} \right\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \limsup_{h \rightarrow 0} \left\| (T_h - S_h)^{[n]} \right\|^{\frac{1}{n}} = 0.$$

The asymptotic (quasinilpotent) equivalence between two families of operators  $\{S_h\}, \{T_h\} \subset B(X)$  is an equivalence relation (i.e. reflexive, symmetric and transitive) on  $L(X)$ . Moreover, if  $\{S_h\}, \{T_h\}$  are two bounded asymptotically equivalent families, then are asymptotically quasinilpotent equivalent.

Let be the sets

$$\begin{aligned} C_b((0, 1], B(X)) &= \\ &= \{ \varphi : (0, 1] \rightarrow B(X) \mid \varphi(h) = T_h \text{ such that } \varphi \text{ is countinuous and bounded} \} = \\ &= \left\{ \{T_h\}_{h \in (0, 1]} \subset B(X) \mid \{T_h\}_{h \in (0, 1]} \text{ is a bounded family, i.e. } \sup_{h \in (0, 1]} \|T_h\| < \infty \right\}. \end{aligned}$$

and

$$\begin{aligned} C_0((0, 1], B(X)) &= \left\{ \varphi \in C_b((0, 1], B(X)) \mid \lim_{h \rightarrow 0} \|\varphi(h)\| = 0 \right\} = \\ &= \left\{ \{T_h\}_{h \in (0, 1]} \subset B(X) \mid \lim_{h \rightarrow 0} \|T_h\| = 0 \right\}. \end{aligned}$$

$C_b((0, 1], B(X))$  is a Banach algebra non-commutative with norm

$$\|\{T_h\}\| = \sup_{h \in (0, 1]} \|T_h\|,$$

and  $C_0((0, 1], B(X))$  is a closed bilateral ideal of  $C_b((0, 1], B(X))$ . Therefore the quotient algebra  $C_b((0, 1], B(X))/C_0((0, 1], B(X))$ , which will be called from now  $B_\infty$ , is also a Banach algebra with quotient norm

$$\|\{\dot{T}_h\}\| = \inf_{\{U_h\}_{h \in (0, 1]} \in C_0((0, 1], B(X))} \|\{T_h\} + \{U_h\}\| = \inf_{\{S_h\}_{h \in (0, 1]} \in \{\dot{T}_h\}} \|\{S_h\}\|.$$

Then

$$\|\{\dot{T}_h\}\| = \inf_{\{S_h\}_{h \in (0, 1]} \in \{\dot{T}_h\}} \|\{S_h\}\| \leq \|\{S_h\}\| = \sup_{h \in (0, 1]} \|S_h\|,$$

for any  $\{S_h\}_{h \in (0, 1]} \in \{\dot{T}_h\}$ . Moreover,

$$\|\{\dot{T}_h\}\| = \inf_{\{S_h\}_{h \in (0, 1]} \in \{\dot{T}_h\}} \|\{S_h\}\| = \inf_{\{S_h\}_{h \in (0, 1]} \in \{\dot{T}_h\}} \sup_{h \in (0, 1]} \|S_h\|.$$

If two bounded families  $\{T_h\}_{h \in (0, 1]}, \{S_h\}_{h \in (0, 1]} \subset B(X)$  are asymptotically equivalent, then  $\lim_{h \rightarrow 0} \|S_h - T_h\| = 0$ , i.e.  $\{T_h - S_h\}_{h \in (0, 1]} \in C_0((0, 1], B(X))$ .

Let  $\{T_h\}_{h \in (0, 1]}, \{S_h\}_{h \in (0, 1]} \in C_b((0, 1], B(X))$  be asymptotically equivalent. Then

$$\limsup_{h \rightarrow 0} \|S_h\| = \limsup_{h \rightarrow 0} \|T_h\|.$$

Since

$$\limsup_{h \rightarrow 0} \|S_h\| \leq \sup_{h \in (0, 1]} \|S_h\|,$$

results that

$$\begin{aligned} \limsup_{h \rightarrow 0} \|S_h\| &= \inf_{\{S_h\}_{h \in (0, 1]} \in \{\dot{T}_h\}} \limsup_{h \rightarrow 0} \|S_h\| \leq \\ &\leq \inf_{\{S_h\}_{h \in (0, 1]} \in \{\dot{T}_h\}} \sup_{h \in (0, 1]} \|S_h\| = \|\{\dot{T}_h\}\|, \end{aligned}$$

for any  $\{S_h\}_{h \in (0, 1]} \in \{\dot{T}_h\}$ .

In particular

$$\lim_{h \rightarrow 0} \lim \|T_h\| \leq \|\{\dot{T}_h\}\| \leq \|\{T_h\}\| = \sup_{h \in (0, 1]} \|T_h\|.$$

**Definition 2.** We say  $\{\dot{S}_h\}, \{\dot{T}_h\} \in B_\infty$  are spectral equivalent if

$$\lim_{n \rightarrow \infty} \left( \left\| \left( \{\dot{S}_h\} - \{\dot{T}_h\} \right)^{[n]} \right\| \right)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left( \left\| \left( \{\dot{T}_h\} - \{\dot{S}_h\} \right)^{[n]} \right\| \right)^{\frac{1}{n}} = 0,$$

$$\text{where } (\{\dot{S}_h\} - \{\dot{T}_h\})^{[n]} = \sum_{k=0}^n (-1)^{n-k} C_n^k \{\dot{S}_h\}^k \{\dot{T}_h\}^{n-k}.$$

$$(\{\dot{S}_h\} - \{\dot{T}_h\})^{[n]} = \sum_{k=0}^n (-1)^{n-k} C_n^k \{\dot{S}_h\}^k \{\dot{T}_h\}^{n-k} = \left\{ \sum_{k=0}^n (-1)^{n-k} C_n^k S_h^k T_h^{n-k} \right\} = \{(S_h - T_h)^{[n]}\}.$$

Therefore  $\{\dot{S}_h\}, \{\dot{T}_h\} \in B_\infty$  are spectral equivalent if

$$\lim_{n \rightarrow \infty} \left\| \left\{ (S_h - T_h)^{[n]} \right\} \right\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left\| \left\{ (T_h - S_h)^{[n]} \right\} \right\|^{\frac{1}{n}} = 0.$$

**Proposition 3.** *If  $\{\dot{S}_h\}, \{\dot{T}_h\} \in B_\infty$  are spectral equivalent, then any  $\{S_h\} \in \{\dot{S}_h\}$  and  $\{T_h\} \in \{\dot{T}_h\}$  are asymptotically spectral equivalent.*

*Proof.* Let  $\{S_h\} \in \{\dot{S}_h\}$  and  $\{T_h\} \in \{\dot{T}_h\}$  be arbitrary. Thus

$$\lim_{n \rightarrow \infty} \overline{\lim_{h \rightarrow 0}} \left\| (S_h - T_h)^{[n]} \right\|^{\frac{1}{n}} \leq \lim_{n \rightarrow \infty} \left\| \left\{ (S_h - T_h)^{[n]} \right\} \right\|^{\frac{1}{n}}.$$

Since  $\{\dot{S}_h\}, \{\dot{T}_h\} \in B_\infty$  are spectral equivalent, by Definition 2 and above relation, it follows that

$$\lim_{n \rightarrow \infty} \overline{\lim_{h \rightarrow 0}} \left\| (S_h - T_h)^{[n]} \right\|^{\frac{1}{n}} = 0.$$

Analogously we can prove that  $\lim_{n \rightarrow \infty} \overline{\lim_{h \rightarrow 0}} \left\| (T_h - S_h)^{[n]} \right\|^{\frac{1}{n}} = 0$ . □

**Proposition 4.** *Let  $\{T_h\}, \{S_h\} \subset B(X)$  be two continuous bounded families. Then  $\lim_{h \rightarrow 0} \|T_h S_h - S_h T_h\| = 0$  if and only if  $\{\dot{S}_h\}\{\dot{T}_h\} = \{\dot{T}_h\}\{\dot{S}_h\}$ .*

*Proof.*  $\lim_{h \rightarrow 0} \|T_h S_h - S_h T_h\| = 0 \Leftrightarrow \{T_h \dot{S}_h\} = \{S_h \dot{T}_h\} \Leftrightarrow \{\dot{S}_h\}\{\dot{T}_h\} = \{\dot{T}_h\}\{\dot{S}_h\}$ . □

**Definition 5.** *We call the resolvent set of a family of operators  $\{S_h\} \in C_b((0, 1], B(X))$  the set*

$$\begin{aligned} r(\{S_h\}) &= \{ \lambda \in \mathbb{C} \mid \exists \{ \mathcal{R}(\lambda, S_h) \} \in C_b((0, 1], B(X)), \lim_{h \rightarrow 0} \|(\lambda I - S_h) \mathcal{R}(\lambda, S_h) - I\| = \\ &= \lim_{h \rightarrow 0} \| \mathcal{R}(\lambda, S_h) (\lambda I - S_h) - I \| = 0 \} \end{aligned}$$

*We call the spectrum of a family of operators  $\{S_h\} \in C_b((0, 1], B(X))$  the set*

$$Sp(\{S_h\}) = \mathbb{C} \setminus r(\{S_h\}).$$

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$r(\{S_h\})$  is an open set of  $C$ . If  $\{S_h\}$  is a bounded family, then  $Sp(\{S_h\})$  is a compact set of  $C$ .

**Remark 6.** i) If  $\lambda \in r(S_h)$  for any  $h \in (0, 1]$ , then  $\lambda \in r(\{S_h\})$ . Therefore  $\bigcap_{h \in (0, 1]} r(S_h) \subseteq r(\{S_h\})$ ;

ii) If  $\lambda \in Sp(\{S_h\})$ , then  $|\lambda| \leq \limsup_{n \rightarrow \infty} \lim_{h \rightarrow 0} \|S_h^n\|^{\frac{1}{n}}$ ;

iii) If  $\|S_h\| < |\lambda|$  for any  $h \in (0, 1]$ , then  $\lambda \in r(\{S_h\})$ ;

iv) If  $\{S_h\}$  is bounded, then  $\{\mathcal{R}(\lambda, S_h)\}$  is also bounded, for every  $\lambda \in r(\{S_h\})$ ;

v) If  $\{S_h\}$  is bounded, then  $\lim_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h)\| \neq 0$ , for every  $\lambda \in r(\{S_h\})$ .

**Proposition 7.** (resolvent equation - asymptotic) Let  $\{S_h\} \subset B(X)$  be a bounded family and  $\lambda, \mu \in r(\{S_h\})$ . Then

$$\lim_{h \rightarrow 0} \|\mathcal{R}(\lambda, S_h) - \mathcal{R}(\mu, S_h) - (\mu - \lambda) \mathcal{R}(\lambda, S_h) \mathcal{R}(\mu, S_h)\| = 0.$$

**Proposition 8.** Let  $\{S_h\} \subset B(X)$  be a bounded family. If  $\lambda \in r(\{S_h\})$  and  $\{\mathcal{R}_i(\lambda, S_h)\} \in C_b((0, 1], B(X))$ ,  $i = \overline{1, 2}$  such that

$$\lim_{h \rightarrow 0} \|(\lambda I - S_h) \mathcal{R}_i(\lambda, S_h) - I\| = \lim_{h \rightarrow 0} \|\mathcal{R}_i(\lambda, S_h) (\lambda I - S_h) - I\| = 0$$

for  $i = \overline{1, 2}$ , then

$$\lim_{h \rightarrow 0} \|\mathcal{R}_1(\lambda, S_h) - \mathcal{R}_2(\lambda, S_h)\| = 0.$$

**Theorem 9.** Let  $\{S_h\} \in C_b((0, 1], B(X))$ . Then

$$Sp(\{\dot{S}_h\}) = Sp(\{S_h\}).$$

**Theorem 10.** If two bounded families  $\{S_h\}, \{T_h\} \subset B(X)$  are asymptotically equivalent, then

$$Sp(\{S_h\}) = Sp(\{T_h\}).$$

## 2 Local Spectrum of a Family of Operators

Let  $\mathcal{O}$  be the set of analytic functions families  $\{f_h\}_{h \in (0, 1]}$  defined on an open complex set with values in a Banach space  $X$ , having property

$$\overline{\lim_{h \rightarrow 0}} \|f_h(\lambda)\| < \infty,$$

for any  $\lambda$  from definition set.

**Definition 11.** A bounded continue family of operators  $\{T_h\} \subset B(X)$  we said to have single-valued extension property, if for any family of analytic functions  $\{f_h\}_{h \in (0, 1]} \in \mathcal{O}$ ,  $f_h : D \rightarrow X$ , where  $D \subset \mathbb{C}$  open, with property

$$\lim_{h \rightarrow 0} \|(\lambda I - T_h) f_h(\lambda)\| \equiv 0,$$

it results  $\lim_{h \rightarrow 0} \|f_h(\lambda)\| \equiv 0$ .

**Remark 12.** Let  $\{S_h\}, \{T_h\} \subset B(X)$  be two bounded continue families of operators asymptotically equivalent. If  $\{S_h\}$  has single-valued extension property, then  $\{T_h\}$  has also single-valued extension property.

*Proof.* Let  $\{f_h\}_{h \in (0,1]} \in \mathcal{O}$  be a family of functions,  $f_h : D \rightarrow X$ , where  $D \subset \mathbb{C}$  open, with  $\lim_{h \rightarrow 0} \|(\lambda I - T_h) f_h(\lambda)\| \equiv 0$ . Then

$$\overline{\lim}_{h \rightarrow 0} \|(\lambda I - S_h) f_h(\lambda)\| = \overline{\lim}_{h \rightarrow 0} \|(\lambda I - S_h - T_h + T_h) f_h(\lambda)\| \leq$$

$$\lim_{h \rightarrow 0} \|(\lambda I - T_h) f_h(\lambda)\| + \overline{\lim}_{h \rightarrow 0} \|(S_h - T_h) f_h(\lambda)\| \leq \lim_{h \rightarrow 0} \|(S_h - T_h)\| \overline{\lim}_{h \rightarrow 0} \|f_h(\lambda)\|,$$

for any  $\lambda \in D$ .

Taking into account  $\{S_h\}, \{T_h\}$  are asymptotically equivalent, it follows

$$\lim_{h \rightarrow 0} \|(\lambda I - T_h) f_h(\lambda)\| \equiv 0.$$

Since  $\{T_h\}$  has single-valued extension property, we obtain  $\lim_{h \rightarrow 0} \|f_h(\lambda)\| \equiv 0$ , thus  $\{S_h\}$  has single-valued extension property.  $\square$

**Definition 13.** Let  $\{T_h\} \subset B(X)$  be a family with single-valued extension property and  $x \in X$ . From now we consider  $r_{\{T_h\}}(x)$  being the set of elements  $\lambda_0 \in \mathbb{C}$  such that there are the analytic functions from  $\mathcal{O}$   $\lambda \mapsto x_h(\lambda)$  defined on an open neighborhood of  $\lambda_0$   $D \subset r_{\{T_h\}}(x)$  with values in  $X$ , for any  $h \in (0, 1]$ , having property

$$\lim_{h \rightarrow 0} \|(\lambda I - T_h) x_h(\lambda) - x\| \equiv 0.$$

$r_{\{T_h\}}(x)$  is called the local resolvent set of  $\{T_h\}$  at  $x$ .

The local spectrum of  $\{T_h\}$  at  $x$  is defined as the set

$$Sp_{\{T_h\}}(x) = \mathbb{C} \setminus r_{\{T_h\}}(x).$$

We also define the local spectral space of  $\{T_h\}$  as

$$X_{\{T_h\}}(a) = \left\{ x \in X \mid Sp_{\{T_h\}}(x) \subset a \right\},$$

for all sets  $a \subset \mathbb{C}$ .

Let be the set

$$X_b((0, 1], X) = \{ \varphi : (0, 1] \rightarrow X \mid \varphi(h) = x_h \text{ such that } \varphi \text{ is continue and bounded} \} =$$

$$= \left\{ \{x_h\}_{h \in (0,1]} \subset X \mid \{x_h\}_{h \in (0,1]} \text{ a bounded sequence, i.e. } \sup_{h \in (0,1]} \|x_h\| < \infty \right\}.$$

and

$$\begin{aligned} X_0((0, 1], X) &= \left\{ \varphi \in X_b((0, 1], X) \mid \lim_{h \rightarrow 0} \|\varphi(h)\| = 0 \right\} = \\ &= \left\{ \{x_h\}_{h \in (0,1]} \subset X \mid \lim_{h \rightarrow 0} \|x_h\| = 0 \right\}. \end{aligned}$$

$X_b((0, 1], X)$  is a Banach space in rapport with norm

$$\|\varphi\| = \sup_{h \in (0, 1]} \|\varphi(h)\| \Leftrightarrow \|\{x_h\}\| = \sup_{h \in (0, 1]} \|x_h\|,$$

and  $X_0((0, 1], X)$  is a closed subspace of  $X_b((0, 1], X)$ . Therefore, the quotient space  $X_b((0, 1], X)/X_0((0, 1], X)$ , which will be called from now  $X_\infty$ , is a Banach space in rapport with quotient norm

$$\begin{aligned} \|\{\dot{x}_h\}\| &= \inf_{\{u_h\}_{h \in (0, 1]} \in X_0((0, 1], X)} \|\{x_h\} + \{u_h\}\| = \\ &= \inf_{\{y_h\}_{h \in (0, 1]} \in \{\dot{x}_h\}} \|\{y_h\}\| = \inf_{\{y_h\}_{h \in (0, 1]} \in \{\dot{x}_h\}} \sup_{h \in (0, 1]} \|y_h\|. \end{aligned}$$

Thus

$$\|\{\dot{x}_h\}\| = \inf_{\{y_h\}_{h \in (0, 1]} \in \{\dot{x}_h\}} \|\{y_h\}\| \leq \|\{y_h\}\| = \sup_{h \in (0, 1]} \|y_h\|,$$

for all  $\{y_h\}_{h \in (0, 1]} \in \{\dot{x}_h\}$ .

Let  $B_\infty = C_b((0, 1], B(X))/C_0((0, 1], B(X))$  and we consider the application  $\Psi$  defines by

$$\left(\{\dot{T}_h\}, \{\dot{x}_h\}\right) \mapsto \{T_h \dot{x}_h\} : B_\infty \times X_\infty \rightarrow X_\infty.$$

**Remark 14.**  $X_\infty$  is a  $B_\infty$  – Banach module in rapport with the above application.

*Proof.* Is the application well defined (i.e. not depending by selection of representatives)?

Let  $\{S_h\}_{h \in (0, 1]} \in \{\dot{T}_h\}$  and  $\{y_h\}_{h \in (0, 1]} \in \{\dot{x}_h\}$ . Then

$$\begin{aligned} \lim_{h \rightarrow 0} \|S_h y_h - T_h x_h\| &= \lim_{h \rightarrow 0} \|S_h y_h - T_h y_h + T_h y_h - T_h x_h\| \leq \\ &\leq \lim_{h \rightarrow 0} \|S_h y_h - T_h y_h\| + \lim_{h \rightarrow 0} \|T_h y_h - T_h x_h\| \leq \\ &\leq \lim_{h \rightarrow 0} \|S_h - T_h\| \lim_{h \rightarrow 0} \|y_h\| + \lim_{h \rightarrow 0} \|T_h\| \lim_{h \rightarrow 0} \|y_h - x_h\| = 0. \end{aligned}$$

Therefore  $\{S_h y_h\}_{h \in (0, 1]} \in \{T_h \dot{x}_h\}$ , for any  $\{S_h\}_{h \in (0, 1]} \in \{\dot{T}_h\}$  and  $\{y_h\}_{h \in (0, 1]} \in \{\dot{x}_h\}$ .

Is  $\Psi$  a bilinear application?

$$\begin{aligned} \Psi\left(\alpha\{\dot{T}_h\} + \beta\{\dot{S}_h\}, \{\dot{x}_h\}\right) &= \Psi\left(\{\alpha T_h + \beta S_h\}, \{\dot{x}_h\}\right) = \\ &= \{(\alpha T_h + \beta S_h) \dot{x}_h\} = \{\alpha T_h \dot{x}_h + \beta S_h \dot{x}_h\} = \\ &= \alpha\{T_h \dot{x}_h\} + \beta\{S_h \dot{x}_h\} = \alpha\Psi\left(\{\dot{T}_h\}, \{\dot{x}_h\}\right) + \beta\Psi\left(\{\dot{S}_h\}, \{\dot{x}_h\}\right), \end{aligned}$$

for any  $\alpha, \beta \in \mathbb{C}$ .

Analogously we can prove that

$$\Psi\left(\{\dot{T}_h\}, \alpha\{\dot{y}_h\} + \beta\{\dot{x}_h\}\right) = \alpha\Psi\left(\{\dot{T}_h\}, \{\dot{y}_h\}\right) + \beta\Psi\left(\{\dot{T}_h\}, \{\dot{x}_h\}\right).$$

Is  $\Psi$  a continue application?

$$\left\|\Psi\left(\{\dot{T}_h\}, \{\dot{x}_h\}\right)\right\| = \left\|\{T_h \dot{x}_h\}\right\| =$$

$$\begin{aligned}
&= \inf_{\{T_h \dot{x}_h\}} \|\{T_h \dot{x}_h\}\| = \inf_{\{T_h \dot{x}_h\}} \sup_{h \in (0,1]} \|T_h \dot{x}_h\| \leq \\
&\leq \inf_{\{T_h \dot{x}_h\}} \sup_{h \in (0,1]} \|T_h\| \|\dot{x}_h\| \leq \inf_{\{\dot{T}_h\}, \{\dot{x}_h\}} \sup_{h \in (0,1]} \|\dot{T}_h\| \|\dot{x}_h\| \leq \\
&\leq \inf_{\{\dot{T}_h\}} \sup_{h \in (0,1]} \|\dot{T}_h\| \inf_{\{\dot{x}_h\}} \sup_{h \in (0,1]} \|\dot{x}_h\| = \|\{\dot{T}_h\}\| \|\{\dot{x}_h\}\|.
\end{aligned}$$

Thus  $\|\Psi(\{\dot{T}_h\}, \{\dot{x}_h\})\| \leq \|\{\dot{T}_h\}\| \|\{\dot{x}_h\}\|$ .

Let  $\{\dot{T}_h\} \in B_\infty$  be fixed. The application  $\{\dot{x}_h\} \mapsto \{T_h \dot{x}_h\}$  is a linear bounded operator on  $X_\infty$ ?

$$\{T_h(\alpha \dot{x}_h + \beta \dot{y}_h)\} = \{\alpha T_h \dot{x}_h + \beta T_h \dot{y}_h\} = \alpha \{T_h \dot{x}_h\} + \beta \{T_h \dot{y}_h\}.$$

In addition, since

$$\|\{T_h \dot{x}_h\}\| \leq \|\{\dot{T}_h\}\| \|\{\dot{x}_h\}\|,$$

it follows the application  $\{\dot{x}_h\} \mapsto \{T_h \dot{x}_h\}$  is a bounded operator.

Therefore,  $B_\infty \subseteq B(X_\infty)$ , where  $B(X_\infty)$  is the algebra of linear bounded operators on  $X_\infty$ .  $\square$

**Definition 15.** We say that  $\{\dot{T}_h\}_{h \in (0,1]} \in B_\infty$  has single-valued extension property if for any analytic function  $f : D_0 \rightarrow X_\infty$ , where  $D_0$  is an open complex set with  $(\lambda \{\dot{I}\} - \{\dot{T}_h\}) f(\lambda) \equiv 0$ , we have  $f(\lambda) \equiv 0$ , where  $0 = \{\dot{0}\} = X_0((0,1], X)$ .

Since  $f(\lambda) \in X_\infty$ , it follows there is  $\{\dot{x}_h(\lambda)\} \in X_\infty$  such that  $f(\lambda) = \{\dot{x}_h(\lambda)\}$ . Then

$$0 \equiv (\lambda \{\dot{I}\} - \{\dot{T}_h\}) f(\lambda) = \{\lambda I - T_h\} \{\dot{x}_h(\lambda)\} = \{(\lambda I - T_h) \dot{x}_h(\lambda)\},$$

i.e.  $\lim_{h \rightarrow 0} \|(\lambda I - T_h) \dot{x}_h(\lambda)\| = 0$ .

**Definition 16.** We say  $\{\dot{T}_h\}_{h \in (0,1]} \in B_\infty$  has the single-valued extension property if for any analytic function  $f : D_0 \rightarrow X_\infty$ , where  $D_0$  is an open complex set with  $\lim_{h \rightarrow 0} \|(\lambda I - T_h) \dot{x}_h(\lambda)\| \equiv 0$  we have  $\lim_{h \rightarrow 0} \|\dot{x}_h(\lambda)\| \equiv 0$ .

The resolvent set of an element  $\{\dot{x}_h\} \in X_\infty$  in rapport with  $\{\dot{T}_h\}_{h \in (0,1]} \in B_\infty$  is  $r_{\{\dot{T}_h\}}(\{\dot{x}_h\}) = \left\{ \lambda_0 \in \mathbb{C} \mid \exists \text{ an analytic function } (\lambda \{\dot{I}\} - \{\dot{T}_h\}) \{\dot{x}_h(\lambda)\} \equiv \{\dot{x}_h\} \right\} =$

$$= \{ \lambda_0 \in \mathbb{C} \mid \exists \text{ an analytic function } \lambda \mapsto \{\dot{x}_h(\lambda)\} : V_{\lambda_0} \rightarrow X_\infty,$$

$$\lim_{h \rightarrow 0} \|(\lambda I - T_h) \dot{x}_h(\lambda) - \dot{x}_h\| \equiv 0 \},$$

when  $V_{\lambda_0}$  is an open neighborhood of  $\lambda_0$ .

Let  $\{\dot{x}\} \in X_\infty$ , where  $\{\dot{x}\} = \{ \{x_h\} \in X_b((0,1], X) \mid \lim_{h \rightarrow 0} \|x_h - x\| = 0 \}$ .

We will call from now

$$X_\infty^0 = \left\{ \{\dot{x}\} \in X_\infty \mid x \in X \right\} \subset X_\infty.$$



Thus

$$r_{\{\dot{T}_h\}}(\{\dot{x}\}) = \{\lambda_0 \in \mathbb{C} \mid \exists \text{ an analytic function } \lambda \mapsto \{x_h(\lambda)\} : V_{\lambda_0} \rightarrow X_\infty, \\ \lim_{h \rightarrow 0} \|(\lambda I - T_h)x_h(\lambda) - x\| \equiv 0\}.$$

**Theorem 17.**  $\{\dot{T}_h\}_{h \in (0,1]} \in B_\infty$  has the single-valued extension property if and only if there is  $\{T_h\} \in \{\dot{T}_h\}$  with single-valued extension property.

*Proof.* Let  $\{f_h\}_{h \in (0,1]} \in \mathcal{O}$ ,  $f_h : D \rightarrow X$ , be a family of analytic functions, when  $D \subset \mathbb{C}$  open, with  $\lim_{h \rightarrow 0} \|(\lambda I - T_h)f_h(\lambda)\| \equiv 0$ .

Since  $\{f_h\}_{h \in (0,1]} \in \mathcal{O}$ , it follows that  $\lim_{h \rightarrow 0} \|f_h(\lambda)\| < \infty$ ,  $\forall \lambda \in D$ , thus  $\{f_h(\lambda)\} \in X_b((0,1], X)$ .

Let  $f : D \rightarrow X_\infty$  be an application defined by  $f(\lambda) = \{\dot{f}_h(\lambda)\}$ . We prove that  $f$  is an analytic function.

Having in view  $\{f_h\}$  are analytic functions on  $D$ , for any  $\lambda_0 \in D$ , we obtain

$$\begin{aligned} \lim_{\lambda \rightarrow \lambda_0} \frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} &= \lim_{\lambda \rightarrow \lambda_0} \frac{\{\dot{f}_h(\lambda)\} - \{\dot{f}_h(\lambda_0)\}}{\lambda - \lambda_0} = \\ &= \lim_{\lambda \rightarrow \lambda_0} \left\{ \frac{\dot{f}_h(\lambda) - \dot{f}_h(\lambda_0)}{\lambda - \lambda_0} \right\} = \left\{ \lim_{\lambda \rightarrow \lambda_0} \frac{\dot{f}_h(\lambda) - \dot{f}_h(\lambda_0)}{\lambda - \lambda_0} \right\}, \end{aligned}$$

for any  $\lambda \in D$ . Therefore,  $f$  is analytic function on  $D$ .

By relation  $\lim_{h \rightarrow 0} \|(\lambda I - T_h)f_h(\lambda)\| \equiv 0$ , i.e.  $(\lambda\{\dot{T}_h\} - \{\dot{T}_h\})f(\lambda) \equiv \{\dot{0}\}$ , since  $\{\dot{T}_h\}$  has the single-valued extension property, it follows that  $f() \equiv \{\dot{0}\}$ , i.e.

$$\lim_{h \rightarrow 0} \|f_h(\lambda)\| \equiv 0.$$

Hence  $\{T_h\}$  has the single-valued extension property.

**Reciprocal:** Let  $\{T_h\}$  has the single-valued extension property. We prove  $\{\dot{T}_h\}$  has also the single-valued extension property.

Let  $f : D \rightarrow X_\infty$  be an analytic application defined by  $f(\lambda) = \{x_h(\lambda)\}$  such that

$$(\lambda\{\dot{T}_h\} - \{\dot{T}_h\})f(\lambda) \equiv \{\dot{0}\}.$$

Then  $\lim_{h \rightarrow 0} \|(\lambda I - T_h)x_h(\lambda)\| \equiv 0$ .

We prove that the applications  $\lambda \mapsto x_h(\lambda) : D \rightarrow X$  are analytical,  $\forall h \in (0,1]$ .

Since  $f$  is analytical function, it follows that

$$f'(\lambda_0) = \lim_{\lambda \rightarrow \lambda_0} \frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} = \lim_{\lambda \rightarrow \lambda_0} \frac{\{x_h(\lambda)\} - \{x_h(\lambda_0)\}}{\lambda - \lambda_0} = \lim_{\lambda \rightarrow \lambda_0} \left\{ \frac{\dot{x}_h(\lambda) - \dot{x}_h(\lambda_0)}{\lambda - \lambda_0} \right\}.$$

Therefore, there is  $\left\{ \lim_{\lambda \rightarrow \lambda_0} \frac{x_h(\lambda) - x_h(\lambda_0)}{\lambda - \lambda_0} \right\} \in X_\infty$  and thus there is  $\lim_{\lambda \rightarrow \lambda_0} \frac{x_h(\lambda) - x_h(\lambda_0)}{\lambda - \lambda_0} \in X$ ,  $\forall h \in (0,1]$ .

Since  $\left(\lambda\{\dot{T}\} - \{\dot{T}_h\}\right)f(\lambda) \equiv \{\dot{0}\}$ , i.e.  $\lim_{h \rightarrow 0} \|(\lambda I - T_h)x_h(\lambda)\| \equiv 0$ , taking into account  $\{\dot{T}_h\}$  has the single-valued extension property, we have  $\lim_{h \rightarrow 0} \|x_h(\lambda)\| \equiv 0$ , i.e.  $\{x_h(\lambda)\} = \{\dot{0}\}$ . Therefore,  $\{\dot{T}_h\}$  has the single-valued extension property.  $\square$

**Proposition 18.** *Let  $\{\dot{T}_h\}_{h \in (0,1]} \in B_\infty$  with the single-valued extension property. Then*

$$r_{\{T_h\}}(x) = r_{\{\dot{T}_h\}}(\{\dot{x}\}),$$

for all  $x \in X$ .

*Proof.* If  $\{\dot{T}_h\}_{h \in (0,1]} \in B_\infty$  has the single-valued extension property, then  $\{T_h\} \in \{\dot{T}_h\}$  has the single-valued extension property (Theorem 17).

Let  $\lambda_0 \in r_{\{T_h\}}(x)$ . Hence there are the analytic functions from  $\mathcal{O}$   $\lambda \mapsto x_h(\lambda)$  defined on an open neighborhood of  $\lambda_0$   $D \subset r_{\{T_h\}}(x)$  with values in  $X$ ,  $\forall h \in (0, 1]$ , having property

$$\lim_{h \rightarrow 0} \|(\lambda I - T_h)x_h(\lambda) - x\| \equiv 0.$$

Similar to proof of Theorem 17, we prove that the application  $f : D \rightarrow X_\infty$  defined by  $f(\lambda) = \{x_h(\lambda)\}$  is analytical. Thus  $\lambda_0 \in r_{\{\dot{T}_h\}}(\{\dot{x}\})$ .

**Reciprocal:** Let

$$\lambda_0 \in r_{\{\dot{T}_h\}}(\{\dot{x}\}) = \{\lambda_0 \in \mathbb{C} \mid \exists \text{ an analytic function } \lambda \mapsto \{x_h(\lambda)\} : V_{\lambda_0} \rightarrow X_\infty,$$

$$\lim_{h \rightarrow 0} \|(\lambda I - T_h)x_h(\lambda) - x\| \equiv 0 \}.$$

Analog proof of Theorem 17, we prove that the applications  $\lambda \mapsto x_h(\lambda) : V_{\lambda_0} \rightarrow X$  are analytical,  $\forall h \in (0, 1]$ . Thus  $\lambda_0 \in r_{\{T_h\}}(x)$ .  $\square$

**Remark 19.** *Let  $\{T_h\} \subset B(X)$  be a continuous bounded family of operators having the single-valued extension property and  $x \in X$ . Then*

- i)  $r(\{T_h\}) \subset r_{\{T_h\}}(x)$ .
- ii)  $X_{\{T_h\}}(a) = X_{\{T_h\}}(Sp\{T_h\} \cap a)$ , for each  $a \subset \mathbb{C}$ .
- iii) Let  $\lambda_0 \in r_{\{T_h\}}(x)$  and the families of holomorphic function from  $\mathcal{O}$   $\lambda \mapsto x_h(\lambda)$  and  $\lambda \mapsto y_h(\lambda)$  defined on  $D$ , an open neighborhood of  $\lambda_0$ , with values in  $X$ , for all  $h \in (0, 1]$ , having properties

$$\lim_{h \rightarrow 0} \|(\lambda I - T_h)x_h(\lambda) - x\| = 0$$

and

$$\lim_{h \rightarrow 0} \|(\lambda I - T_h)y_h(\lambda) - x\| = 0,$$

for each  $\lambda \in D$ . Then

$$\lim_{h \rightarrow 0} \|x_h(\lambda) - y_h(\lambda)\| = 0,$$

for each  $\lambda \in D$ .

iv) If  $\{T_h\}, \{S_h\} \in C_b((0, 1], B(X))$  are asymptotically equivalent, then

$$r_{\{T_h\}}(x) = r_{\{S_h\}}(x), \quad \forall x \in X.$$

*Proof.* i) By Proposition 18 we have

$$r_{\{\dot{T}_h\}}(\{\dot{x}\}) = r_{\{T_h\}}(x), \quad \forall x \in X.$$

Moreover, by Theorem 9, we know that

$$r(\{\dot{T}_h\}) = r(\{T_h\}).$$

Combing the above relations, we obtain

$$r(\{T_h\}) = r(\{\dot{T}_h\}) \subset r_{\{\dot{T}_h\}}(\{\dot{x}\}) = r_{\{T_h\}}(x), \quad \forall x \in X.$$

ii) By i) it results

$$Sp_{\{T_h\}}(x) \subset Sp(\{T_h\}).$$

Therefore  $x \in X_{\{T_h\}}(a)$  if and only if

$$Sp_{\{T_h\}}(x) \subset a \bigcap Sp(\{T_h\}),$$

i.e.  $x \in X_{\{T_h\}}(a \bigcap Sp(\{T_h\}))$ .

iii) By Definition 13., it results that the analytic functions  $\lambda \mapsto x_h(\lambda)$  are defined on an open neighborhood of  $\lambda_0$   $D_1 \subset r(\{T_h\})$  with values in  $X$  and the analytic functions  $\lambda \mapsto y_h(\lambda)$  are defined on an open neighborhood of  $\lambda_0$   $D_2 \subset r(\{T_h\})$  on  $X$ .

Let  $D \subset D_1 \cap D_2 \subset r(\{T_h\})$  be an open neighborhood of  $\lambda_0$ .

Since

$$\lim_{h \rightarrow 0} \|(\lambda I - T_h)x_h(\lambda) - x\| = 0$$

and

$$\lim_{h \rightarrow 0} \|(\lambda I - T_h)y_h(\lambda) - x\| = 0,$$

for each  $\lambda \in D$ , thus

$$\lim_{h \rightarrow 0} \|(\lambda I - T_h)x_h(\lambda) - (\lambda I - T_h)y_h(\lambda)\| = \lim_{h \rightarrow 0} \|(\lambda I - T_h)(x_h(\lambda) - y_h(\lambda))\| = 0,$$

for each  $\lambda \in D$ .

Having in view that the families of functions  $\lambda \mapsto x_h(\lambda)$  and  $\lambda \mapsto y_h(\lambda)$  are analytical on  $D$ , hence the functions  $\lambda \mapsto x_h(\lambda) - y_h(\lambda)$  are analytical. Since  $\{T_h\}$  has the single-valued extension property, it follows that

$$\lim_{h \rightarrow 0} \|x_h(\lambda) - y_h(\lambda)\| = 0,$$

for all  $\lambda \in D$ .

iv) Let  $\lambda_0 \in r_{\{T_h\}}(x)$ . Then there is a family of functions  $\{x_h\}$  from  $\mathcal{O}$ , with the property

$$\lim_{h \rightarrow 0} \|(\lambda I - T_h) x_h(\lambda) - x\| \equiv 0.$$

Thus

$$\begin{aligned} \overline{\lim}_{h \rightarrow 0} \|(\lambda I - S_h) x_h(\lambda) - x\| &= \overline{\lim}_{h \rightarrow 0} \|(\lambda I - S_h - T_h + T_h) x_h(\lambda) - x\| \leq \\ &\leq \lim_{h \rightarrow 0} \|(\lambda I - T_h) x_h(\lambda) - x\| + \overline{\lim}_{h \rightarrow 0} \|(S_h - T_h) x_h(\lambda)\| \leq \\ &\leq \lim_{h \rightarrow 0} \|S_h - T_h\| \overline{\lim}_{h \rightarrow 0} \|x_h(\lambda)\|. \end{aligned}$$

Since  $\{T_h\}$ ,  $\{S_h\}$  are asymptotically equivalent, by above relation it follows that

$$\lim_{h \rightarrow 0} \|(\lambda I - S_h) x_h(\lambda) - x\| \equiv 0.$$

Therefore  $\lambda_0 \in r_{\{S_h\}}(x)$ . □

**Proposition 20.** *Let  $\{T_h\} \subset B(X)$  be a continuous bounded family of operators having the single-valued extension property. Then*

- i) *For any  $a \subset b$  we have  $X_{\{T_h\}}(a) \subset X_{\{T_h\}}(b)$ ;*
- ii)  *$X_{\{T_h\}}(a)$  is a linear sub-space of  $X$ ,  $\forall a \subset \mathbb{C}$ ;*
- iii)  *$\left\{ \{x\} \in X_\infty \mid x \in X_{\{T_h\}}(a) \right\} = X_\infty^0 \cap X_{\{T_h\}}(a), \forall a \subset \mathbb{C}$ .*

*Proof.* i) Let  $a, b \subset \mathbb{C}$  such that  $a \subset b$  and  $x \in X_{\{T_h\}}(a)$ . Then  $Sp_{\{T_h\}}(x) \subset a$ , and thus  $Sp_{\{T_h\}}(x) \subset b$ . Therefore  $x \in X_{\{T_h\}}(b)$ .

ii) Let  $x, y \in X_{\{T_h\}}(a)$  and  $\alpha, \beta \in \mathbb{C}$ . In addition, for any  $\lambda_0 \in r_{\{T_h\}}(x) \cap r_{\{T_h\}}(y)$  there are the analytic functions families  $\{x_h\}$  and  $\{y_h\}$  defined on an open neighborhood  $D$  of  $\lambda_0$  such that

$$\lim_{h \rightarrow 0} \|(\lambda I - T_h) x_h(\lambda) - x\| = 0$$

and

$$\lim_{h \rightarrow 0} \|(\lambda I - T_h) y_h(\lambda) - y\| = 0,$$

for each  $\lambda \in D$ .

Let  $z_h(\lambda) = \alpha x_h(\lambda) + \beta y_h(\lambda)$ , for any  $\lambda \in D$  and  $h \in (0, 1]$ . Since  $\{x_h\}$  and  $\{y_h\}$  are analytic functions families on  $D$ , it follows that  $\{z_h\}$  is also an analytic functions family on  $D$  and more

$$\begin{aligned} \lim_{h \rightarrow 0} \|(\lambda I - T_h) z_h(\lambda) - (\alpha x + \beta y)\| &\leq \\ &\leq |\alpha| \lim_{h \rightarrow 0} \|(\lambda I - T_h) x_h(\lambda) - x\| + |\beta| \lim_{h \rightarrow 0} \|(\lambda I - T_h) y_h(\lambda) - y\| = 0, \end{aligned}$$

for each  $\lambda \in D$ .

Therefor  $\lambda_0 \in r_{\{T_h\}}(\alpha x + \beta y)$  and

$$r_{\{T_h\}}(x) \cap r_{\{T_h\}}(y) \subset r_{\{T_h\}}(\alpha x + \beta y).$$

Moreover

$$Sp_{\{T_h\}}(\alpha x + \beta y) \subset Sp_{\{T_h\}}(x) \bigcup Sp_{\{T_h\}}(y).$$

Since  $x, y \in X_{\{T_h\}}(a)$ , i.e.  $Sp_{\{T_h\}}(x) \subset a$  and  $Sp_{\{T_h\}}(y) \subset a$ , by above relation, it follows that

$$Sp_{\{T_h\}}(\alpha x + \beta y) \subset a,$$

hence  $\alpha x + \beta y \in X_{\{T_h\}}(a)$ .

iii) Since by Proposition 18 we have  $(r_{\{T_h\}}(x) = r_{\{\dot{T}_h\}}(\{\dot{x}\}))$ , it follows that  $x \in X_{\{T_h\}}(a)$  if and only if  $\{\dot{x}\} \in X_{\{\dot{T}_h\}}(a)$ . Hence

$$\begin{aligned} \left\{ \{\dot{x}\} \in X_\infty \mid x \in X_{\{T_h\}}(a) \right\} &= \left\{ \{\dot{x}\} \in X_\infty \mid Sp_{\{T_h\}}(x) \subset a \right\} = \\ &= \left\{ \{\dot{x}\} \in X_\infty \mid Sp_{\{\dot{T}_h\}}(\{\dot{x}\}) \subset a \right\} = X_\infty^0 \cap X_{\{\dot{T}_h\}}(a). \end{aligned}$$

□

**Theorem 21.** Let  $\{S_h\}, \{T_h\} \subset B(X)$  be two continuous bounded families of operators having the single-valued extension property, such that  $\lim_{h \rightarrow 0} \|T_h S_h - S_h T_h\| = 0$ . If  $\{S_h\}, \{T_h\}$  are asymptotically spectral equivalent, then

$$Sp_{\{T_h\}}(x) = Sp_{\{S_h\}}(x), \quad \forall x \in X.$$

*Proof.* Since  $\{S_h\}, \{T_h\}$  have the single-valued extension property, by Theorem 17 it results that  $\{T_h\}_{h \in (0,1]}, \{\dot{S}_h\}_{h \in (0,1]} \in B_\infty$  have the single-valued extension property.

If  $\{S_h\}, \{T_h\}$  are asymptotically spectral equivalent, by Proposition 3 have that  $\{\dot{T}_h\}_{h \in (0,1]}, \{\dot{S}_h\}_{h \in (0,1]}$  are spectral equivalent. Moreover, we obtain that for any  $\{\dot{T}_h\}_{h \in (0,1]}, \{\dot{S}_h\}_{h \in (0,1]} \in B_\infty$  have the single-valued extension property and being spectral equivalent, it follows that

$$Sp_{\{\dot{T}_h\}}(\{\dot{x}\}) = Sp_{\{\dot{S}_h\}}(\{\dot{x}\}),$$

for any  $x \in X$ .

Therefore, applying Proposition 18, we have

$$Sp_{\{T_h\}}(x) = Sp_{\{\dot{T}_h\}}(\{\dot{x}\}) = Sp_{\{\dot{S}_h\}}(\{\dot{x}\}) = Sp_{\{S_h\}}(x), \quad \forall x \in X.$$

□

**Remark 22.** Let  $\{S_h\}, \{T_h\} \subset B(X)$  be two continuous bounded families of operators having the single-valued extension property, such that  $\lim_{h \rightarrow 0} \|T_h S_h - S_h T_h\| = 0$ . If  $\{S_h\}, \{T_h\}$  are asymptotically spectral equivalent, then

$$X_{\{T_h\}}(a) = X_{\{S_h\}}(a),$$

for any  $a \in \mathbb{C}$ .

*Proof.* Since  $\{S_h\}$ ,  $\{T_h\}$  are asymptotically spectral equivalent, by Theorem 21, it follows that  $Sp_{\{T_h\}}(x) = Sp_{\{S_h\}}(x)$ , for all  $x \in X$ . Then, for any  $x \in X_{\{T_h\}}(a)$ , i.e.  $Sp_{\{T_h\}}(x) \subset a$ , it results that  $x \in X_{\{S_h\}}(a)$ , thus

$$X_{\{T_h\}}(a) \subseteq X_{\{S_h\}}(a).$$

Analog, we can show that  $X_{\{S_h\}}(a) \subseteq X_{\{T_h\}}(a)$ . □

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